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New fixed point results in rectangular metric space and application to fractional calculus

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Abstract

In this paper, we introduce $\alpha-\psi$ type contractive mapping in rectangular metric space satisfying certain admissibility conditions and prove a fixed point result for such mapping in complete and Hausdorff rectangular metric space. Some examples are given to justify our result. Also we have shown that the existence of solution of a nonlinear fractional differential equation can be guaranteed, as an application of our result.

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Dedicated to professor H. M. Srivastava.

1 Introduction

There are several generalisations of metric space in the literature which were intended to extend fixed point results already known in ordinary metric space. One such generalisation was introduced by Branciari in 2000. He gave the notation of a generalized metric space (or rectangular metric space), in which triangular inequality is replaced by quadrilateral inequality. It differs from ordinary metric as

1. Rectangular metric need not to be continuous
2. In rectangular metric space, a convergent sequence need not to be a Cauchy sequence.
3. Rectangular metric space need not to be a Hausdorff space.

Samet et al. [6] introduced the concept of $\alpha-\psi-$ contractive mapping and proved fixed point theorems for such mappings. In [4], Karapınar gave contractive conditions to obtain the existence and uniqueness of fixed point of $\alpha-\psi$ contraction mappings in the rectangular metric spaces. Salimi et al. [7] introduced modified $\alpha-\psi$ contractive mappings and obtained some fixed point theorems in complete metric space. Alsulami et al. [1] established some fixed point theorems for $\alpha-\psi-$rational type contractive mappings in rectangular metric space.

Following this direction of research, in this paper we prove a new fixed point theorem for the admissible mappings in the context of rectangular metric spaces which enable us to establish the existence of solution of a nonlinear fractional differential equation satisfying integral boundary conditions.

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2 Mathematical Preliminaries

Definition 2.1. [3] Let X be a nonempty set and let $d : X \times X \to [0, \infty)$ satisfy the following conditions for all $x, y \in X$ and all distinct $u, v \in X$ each of them different from x and y.

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (quadrilateral inequality).

Then the function d is called a rectangular metric and the pair (X, d) is called a rectangular metric spaces (for short RMS).

In the literature, some researcher use the notation of generalized metric space (or in short g.m.s) for rectangular metric space.

Definition 2.2. [3] Let (X, d) be a rectangular metric space and $\{x_n\}$ be a sequence in X.

1. $\{x_n\}$ is called (g.m.s) convergent to a limit x if and only if $d(x_n, x) \to 0$ as $n \to \infty$,
2. $\{x_n\}$ is called a (g.m.s) Cauchy sequence if and only if for every $\varepsilon > 0$ there exists positive integer $N(\varepsilon)$ such that $d(x_n, x_m) < \varepsilon$ for all $m > n \geq N(\varepsilon)$,
3. A rectangular metric space (X, d) is called complete if every (g.m.s) Cauchy sequence is a (g.m.s) convergent.

Let Ψ be the family of all functions $\psi : [0, \infty) \to [0, \infty)$ such that ψ is nondecreasing and $\sum \psi^n(t) < \infty$ for each $t > 0$.

If $\psi \in \Psi$, then for each $t > 0$ $\lim\limits_{n\to\infty} \psi^n(t) = 0$ implies $\psi(t) < t$.

Definition 2.3. [7] Let T be a self mapping on a metric space (X, d) and let $\alpha, \eta : X \times X \to [0, \infty)$ be two functions. T is called an $\alpha-$admissible mapping with respect to η if $\alpha(x, y) \geq \eta(x, y)$ implies that $\alpha(Tx, Ty) \geq \eta(Tx, Ty)$ for all $x, y \in X$.
If $\eta(x, y) = 1$ for all $x, y \in X$, then T is called α-admissible mapping.

3 Main Results

Definition 3.1. Let $\{x_n\}$ be any sequence in X with $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$, for all $n \geq N$ and $x_n \to x$ as $n \to \infty$, implies that $\alpha(x_n, x) \geq \eta(x_n, x)$, for all $n \geq N$, then we say that X is $\alpha-$regular with respect to η.

Theorem 3.2. Let (X, d) be a Hausdorff and complete RMS, and let $T : X \to X$ be an $\alpha-$admissible mapping with respect to η. Assume that there exists continuous function $\psi \in \Psi$ such that

$$x, y \in X,\ \alpha(x, y) \geq \eta(x, y) \text{ implies } d(Tx, Ty) \leq \psi(M(x, y)). \tag{3.1}$$

where

$$M(x, y) = \max \left\{ \begin{array}{c} d(x, y), d(x, Tx), d(y, Ty), \\ \frac{1}{1+d(x,y)}[d(x, Tx)d(y, Ty)], \frac{1}{1+d(Tx,Ty)}[d(x, Tx)d(y, Ty)] \end{array} \right\}.$$

Also, suppose that the following assertions are hold;

1. there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$,
2. for all $x, y, z \in X$, $\alpha(x, y) \geq \eta(x, y)$ and $\alpha(y, z) \geq \eta(y, z)$ implies $\alpha(x, z) \geq \eta(x, z)$
3. either T is continuous or X is $\alpha-$regular with respect to η.

Then T has a periodic point $a \in X$ and if $\alpha(a, Ta) \geq \eta(a, Ta)$ holds for each periodic point then T has a fixed point. Moreover, if for all $x, y \in F(T)$, we have $\alpha(x, y) \geq \eta(x, y)$, then the fixed point is unique.

Proof. By (1), there exists $x_0 \in X$ such that

$$\alpha\left(x_0, Tx_0\right) \geq \eta\left(x_0, Tx_0\right) \tag{3.2}$$

Define a sequence $\{x_n\}$ in X by $x_n = Tx_{n-1} = T^n x_0$ for $n = 1, 2, 3 \ldots$. If $x_{n_0+1} = x_{n_0}$ for some $n_0 \in \mathbb{N}$, then $x = x_{n_0}$ is a fixed point of T. Suppose that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Note that T is an $\alpha-$admissible mapping with respect to η and consider (3.2), we deduce that

$$\begin{aligned} \alpha\left(x_1, x_2\right) &= \alpha\left(Tx_0, T^2x_0\right) \\ &\geq \eta\left(Tx_0, T^2x_0\right) \\ &= \eta\left(x_1, x_2\right). \end{aligned}$$

Continuing this process, we get

$$\alpha\left(x_n, x_{n+1}\right) \geq \eta\left(x_n, x_{n+1}\right), \text{ for all } n \geq \mathbb{N}. \tag{3.3}$$

Here we give the proof by steps
Step 1: The sequence $\{d\left(x_n, x_{n+1}\right)\}$ is non-increasing and $d\left(x_n, x_{n+1}\right) \to 0$ as $n \to \infty$. Indeed, we have

$$d\left(x_n, x_{n+1}\right) = d\left(Tx_{n-1}, Tx_n\right) \leq \psi\left(M\left(x_{n-1}, x_n\right)\right) \tag{3.4}$$

where

$$\begin{aligned} M\left(x_{n-1}, x_n\right) &= \max\left\{ \begin{array}{c} d\left(x_{n-1}, x_n\right), d\left(x_{n-1}, x_n\right), d\left(x_n, x_{n+1}\right), \\ \frac{1}{1+d(x_{n-1}, x_n)}\left[d\left(x_{n-1}, x_n\right) d\left(x_n, x_{n+1}\right)\right], \\ \frac{1}{1+d(x_n, x_{n+1})}\left[d\left(x_{n-1}, x_n\right) d\left(x_n, x_{n+1}\right)\right] \end{array} \right\} \\ &\leq \max\left\{ \begin{array}{c} d\left(x_{n-1}, x_n\right), d\left(x_n, x_{n+1}\right), \\ \frac{1}{d(x_{n-1}, x_n)}\left[d\left(x_{n-1}, x_n\right) d\left(x_n, x_{n+1}\right)\right], \\ \frac{1}{d(x_n, x_{n+1})}\left[d\left(x_{n-1}, x_n\right) d\left(x_n, x_{n+1}\right)\right] \end{array} \right\} \\ &= \max\left\{ d\left(x_{n-1}, x_n\right), d\left(x_n, x_{n+1}\right) \right\} \end{aligned}$$

In this instance, we have two cases.
Case 1: If $M(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$, then from definition of ψ, we get the following contradiction

$$d(x_n, x_{n+1}) \leq d(Tx_{n-1}, Tx_n) \leq \psi(M(x_{n-1}, x_n)) = \psi(d(x_n, x_{n+1})) < d(x_n, x_{n+1})$$

Case 2: If $M(x_{n-1}, x_n) = d(x_{n-1}, x_n)$ then

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \psi(M(x_{n-1}, x_n)) = \psi(d(x_{n-1}, x_n)) < d(x_{n-1}, x_n)$$

Hence we get $\{d(x_n, x_{n+1})\}$ is a decreasing sequence of non-negative real numbers, that is,

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n) \tag{3.5}$$

Hence, there is $r \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r.$$

Also, we have

$$\lim_{n\to\infty} M(x_{n-1}, x_n) \leq r$$

Taking lim as $n \to \infty$ in (3.4), we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) \leq \lim_{n\to\infty} \psi(M(x_{n-1}, x_n))$$

As ψ is continuous and non-decreasing, we have

$$r \leq \psi(r) \tag{3.6}$$

since $\psi \in \Psi$ this implies that $r = 0$, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Step 2: $d(x_n, x_{n+2}) \to 0$ as $n \to \infty$. Indeed, we have

$$d(x_n, x_{n+2}) = d(Tx_{n-1}, Tx_{n+1}) \leq \psi(M(x_{n-1}, x_{n+1})) \tag{3.7}$$

where

$$\begin{aligned} M(x_{n-1}, x_{n+1}) &= \max\left\{\begin{array}{c} d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2}), \\ \frac{1}{1+d(x_{n-1},x_{n+1})}[d(x_{n-1}, x_n)\, d(x_{n+1}, x_{n+2})], \\ \frac{1}{1+d(x_n,x_{n+2})}[d(x_{n-1}, x_n)\, d(x_{n+1}, x_{n+2})] \end{array}\right\} \\ &\leq \max\left\{\begin{array}{c} d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n+1}, x_{n+2}), \\ [d(x_{n-1}, x_n)\, d(x_{n+1}, x_{n+2})] \\ [d(x_{n-1}, x_n)\, d(x_{n+1}, x_{n+2})] \end{array}\right\} \end{aligned}$$

Thanks to (3.5), we have some obliteration, that is, we have

$$\begin{aligned} M(x_{n-1}, x_{n+1}) &\leq \max\left\{\begin{array}{c} d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), \\ [d(x_{n-1}, x_n)]^2, [d(x_{n-1}, x_n)]^2 \end{array}\right\} \\ &\leq \max\left\{d(x_{n-1}, x_{n+1}), d(x_{n-1}, x_n), d(x_{n-1}, x_n)^2\right\}, \text{ for each } n \geq \mathbb{N}. \end{aligned}$$

As a convenience, let

$$a_n := d(x_{n-1}, x_{n+1}) \text{ and } b_n := d(x_{n-1}, x_n).$$

Thus,

$$M(x_{n-1}, x_{n+1}) \leq \max\left\{a_n, b_n, [b_n]^2\right\}, \text{ for each } n \geq \mathbb{N}.$$

In here we have three cases. If $M(x_{n-1}, x_{n+1}) \leq b_n$ or $M(x_{n-1}, x_{n+1}) \leq [b_n]^2$. Since $b_n \to 0$ as $n \to \infty$, then from (3.7) we have

$$\begin{aligned} \lim_{n\to\infty} d(x_n, x_{n+2}) &\leq \lim_{n\to\infty} \psi(M(x_{n-1}, x_{n+1})) \\ &= 0 \end{aligned}$$

If $M(x_{n-1}, x_{n+1}) \leq a_n$, then we see that

$$d(x_n, x_{n+2}) \leq \psi(d(x_{n-1}, x_{n+1})) < d(x_{n-1}, x_{n+1})$$

Thus, the sequence $\{d(x_n, x_{n+2})\}$ is a decreasing sequence of non-negative real numbers and hence

$$d(x_n, x_{n+2}) \to 0 \text{ as } n \to \infty. \tag{3.8}$$

Step 3: Now, we claim that T has a periodic point. Assume that T has no periodic point; then $\{x_n\}$ is a sequence of distinct points, that is, $x_n \neq x_m$ for all $m \neq n$. In this case we will get that $\{x_n\}$ is a g.m.s Cauchy sequence. If not, then there exists some $\varepsilon > 0$ for which we can find two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$, $m(k) > n(k) > k$ for each $k \geq \mathbb{N}$ such that

$$d\left(x_{m(k)}, x_{n(k)}\right) \geq \varepsilon \text{ and } d\left(x_{m(k)-1}, x_{n(k)}\right) < \varepsilon$$

Note that $\{x_n\}$ is a sequence of distinct points, then thanks to rectangular inequality, we have

$$\begin{aligned} \varepsilon &\leq d\left(x_{m(k)}, x_{n(k)}\right) \\ &\leq d\left(x_{m(k)}, x_{m(k)-2}\right) + d\left(x_{m(k)-2}, x_{m(k)-1}\right) + d\left(x_{m(k)-1}, x_{n(k)}\right) \\ &\leq d\left(x_{m(k)}, x_{m(k)-2}\right) + d\left(x_{m(k)-2}, x_{m(k)-1}\right) + \varepsilon. \end{aligned}$$

Thus,

$$\lim_{k\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \varepsilon.$$

Using rectangular inequality we have $\lim_{k\to\infty} d(x_{m(k)-1}, x_{m(k)-1}) = \varepsilon$. Using (3.3) and contractive condition we have

$$\begin{aligned} \varepsilon &\leq d\left(x_{m(k)}, x_{n(k)}\right) \\ &\leq \psi\left(M\left(x_{m(k)-1}, x_{n(k)-1}\right)\right) \end{aligned} \tag{3.9}$$

where

$$M\left(x_{m(k)-1}, x_{n(k)-1}\right) = \max \left\{ \begin{array}{c} d\left(x_{m(k)-1}, x_{n(k)-1}\right), d\left(x_{m(k)-1}, x_{m(k)}\right), d\left(x_{n(k)-1}, x_{n(k)}\right) \\ \frac{1}{1+d\left(x_{m(k)-1}, x_{n(k)-1}\right)} \left[d\left(x_{m(k)-1}, x_{m(k)}\right) d\left(x_{n(k)-1}, x_{n(k)}\right)\right], \\ \frac{1}{1+d\left(x_{m(k)}, x_{n(k)}\right)} \left[d\left(x_{m(k)-1}, x_{m(k)}\right) d\left(x_{n(k)-1}, x_{n(k)}\right)\right], \end{array} \right\}$$

and

$$\lim_{k\to\infty} M\left(x_{m(k)-1}, x_{n(k)-1}\right) = \varepsilon$$

Again taking limit as $n \to \infty$ and since ψ is continuous we get

$$\varepsilon \leq \psi(\varepsilon) < \varepsilon. \tag{3.10}$$

It is clear that (3.10) is a contradiction. Hence, we conclude that $\{x_n\}$ is a g.m.s Cauchy sequence. Since, (X, d) is complete, then there exists $z \in X$ such that $\{x_n\}$ is g.m.s convergent to z. In here we have two cases:
Case 1: Let T be continuous, then

$$x_{n+1} = Tx_n \to Tz \text{ as } n \to \infty. \tag{3.11}$$

and since X is Hausdorff we have $z = Tz$.
Case 2: Let X be $\alpha-$regular with respect to η. From (3.3), we have $\alpha(x_n, z) \geq \eta(x_n, z)$, for all $n \geq \mathbb{N}$. This implies that

$$d(Tx_n, Tz) \leq \psi(M(x_n, z)) \tag{3.12}$$

where

$$M(x_n, z) = \max\left\{\begin{array}{c} d(x_n, z), d(x_n, x_{n+1}), d(z, Tz), \\ \frac{1}{1+d(x_n,z)}[d(z, Tz).d(x_n, x_{n+1})], \\ \frac{1}{1+d(x_{n+1},Tz)}[d(z, Tz).d(x_n, x_{n+1})] \end{array}\right\}$$

Since $\{x_n\} \to z$ as $n \to \infty$, then we have

$$\lim_{n\to\infty} M(x_n, z) \leq d(z, Tz) \tag{3.13}$$

Taking limit as $n \to \infty$ in (3.12) and using continuity of ψ we get

$$d(z, Tz) \leq \psi(d(z, Tz))$$

which implies $d(z, Tz) = 0$ and so $z = Tz$. Hence T has a Periodic point.
Step 4: We claim that T has a fixed point. Note that there exists $a \in X$ such that $a = T^p a$. It is clear that $a \in X$ is a fixed point of T for $p = 1$. We will prove that $\vartheta = T^{p-1}a$ is a fixed point of T in case of $p > 1$. If possible, assume the contrary, that is, let $T^{p-1}a \neq T^p a$. As $\alpha(a, Ta) \geq \eta(a, Ta)$ and T is α admissible w.r.t. η we have $\alpha(T^n a, T^n Ta) \geq \eta(T^n a, T^n Ta)$ for all $n \in \mathbb{N}$. Thus from the contractive inequality (3.1) we have

$$\begin{aligned} d(a, Ta) &= d\left(T^p a, T^{p+1}a\right) \\ &\leq \psi\left(M\left(T^{p-1}a, T^p a\right)\right) \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} M\left(T^{p-1}a, T^p a\right) &= \max\left\{\begin{array}{c} d\left(T^{p-1}a, T^p a\right), d\left(T^{p-1}a, T^p a\right), d\left(T^p a, T^{p+1}a\right), \\ \frac{1}{1+d(T^{p-1}a,T^p a)}\left[d\left(T^{p-1}a, T^p a\right) d\left(T^p a, T^{p+1}a\right)\right], \\ \frac{1}{1+d(T^p a,T^{p+1}a)}\left[d\left(T^{p-1}a, T^p a\right) d\left(T^p a, T^{p+1}a\right)\right] \end{array}\right\} \\ &= \max\{d(T^{p-1}a, T^p a), d(T^p a, T^{p+1}a)\} \end{aligned}$$

Arguing as in step 1, if $M(T^{p-1}a, T^p a) = d(T^p a, T^{p+1}a)$ we get contradiction.

$$\text{Therefore, } d(a, Ta) = d\left(T^p a, T^{p+1}a\right) \leq \psi\left(d\left(T^{p-1}a, T^p a\right)\right) < d\left(T^{p-1}a, T^p a\right) \tag{3.15}$$

Continuing the process (3.15), we have the following contradiction;

$$d(a,Ta) < d\left(T^{p-1}a, T^{p}a\right) < d\left(T^{p-2}a, T^{p-1}a\right) < \cdots < d(a,Ta).$$

Hence, the assumption that $\vartheta = T^{p-1}a$ is not a fixed point of T is not true. Consequently, T has a fixed point.

Step 5: Now, we show that the fixed point is unique. If possible, let $\vartheta, \vartheta' \in X$ be distinct fixed point of T. Then $\alpha(v,v') \geq \eta(v,v')$. From the contractive inequality (3.1), we have

$$d(\vartheta,\vartheta') = d(T\vartheta, T\vartheta') \leq \psi\left(M(\vartheta,\vartheta')\right) \tag{3.16}$$

where

$$\begin{aligned} M(\vartheta,\vartheta') &= \max\left\{ \begin{array}{c} d(\vartheta,\vartheta'), d(\vartheta,T\vartheta), d(\vartheta',T\vartheta'), \\ \frac{1}{1+d(\vartheta,\vartheta')} d(\vartheta,T\vartheta)\, d(\vartheta',T\vartheta'), \frac{1}{1+d(T\vartheta,T\vartheta')} d(\vartheta,T\vartheta)\, d(\vartheta',T\vartheta') \end{array} \right\} \\ &= d(\vartheta,\vartheta'). \end{aligned}$$

Return to (3.16), we have

$$d(\vartheta,\vartheta') = d(T\vartheta,T\vartheta') \leq \psi\left(d(\vartheta,\vartheta')\right) < d(\vartheta,\vartheta') \tag{3.17}$$

$\psi \in \Psi$ implies $d(\vartheta,\vartheta') = 0$, that is, fixed point is unique. Q.E.D.

Example 3.3. Let $X = \{1,3,5,7\}$ and $T : X \to X$ be defined as

$$T(1) = 1, T(3) = 5, T(5) = 7, T(7) = 1.$$

Let $\alpha, \eta : X \times X \to [0,\infty)$ given by

$$\alpha(x,y) = 2 \quad \text{and} \quad \eta(x,y) = 1$$

for all $x, y \in X \times X$. Let $d : X \times X \to [0,\infty)$ be defined as

$$d(x,y) = d(y,x) \text{ and } d(x,y) = 0 \text{ iff } x = y$$

$$d(1,3) = 6, \quad d(1,5) = 2, \quad d(1,7) = 1, \quad d(3,5) = 3, \quad d(3,7) = 2, \quad d(5,7) = 2.$$

Then (X,d) is rectangular metric space. Also it is Hausdorff as
$B_1(1) = \{y \in X : d(y,1) < 1\} = \{1\}$, $B_2(3) = \{y \in X : d(y,3) < 2\} = \{3\}$,
$B_2(5) = \{y \in X : d(y,5) < 2\} = \{5\}$, $B_2(7) = \{y \in X : d(y,7) < 2\} = \{7\}$ are disjoint.
For our convenience we use following symbols

$$A = d(x,y), \quad B = d(x,Tx), \quad C = d(y,Ty)$$

$$D = \frac{d(x,Tx)d(y,Ty)}{1+d(x,y)} \text{ and } E = \frac{d(x,Tx)d(y,Ty)}{1+d(Tx,Ty)}$$

	$x=1,y=3$ $Tx=1,Ty=5$	$x=1,y=5$ $Tx=1,Ty=7$	$x=1,y=7$ $Tx=1,Ty=1$	$x=3,y=5$ $Tx=5,Ty=7$	$x=3,y=7$ $Tx=5,Ty=1$	$x=5,y=7$ $Tx=7,Ty=1$
A	6	2	1	3	1	2
B	0	0	0	3	3	2
C	3	2	1	2	3	1
D	0	0	0	1.5	3	0.6666
E	0	0	0	2	3	1
$d(Ty,Ty)$	2	1	0	2	2	1
$M(x,y)$	6	2	1	3	3	2

If we define $\psi : [0,\infty) \to [0,\infty)$ as $\psi(t) = \frac{4}{5}t$ then from the above table it is clear that $d(Tx,Ty) \leq \psi(M(x,y))$ holds whenever $\alpha(x,y) \geq \eta(x,y)$. Thus all the conditions of the above theorem are satisfied with $\psi(t) = \frac{4}{5}t$. Hence T has a unique fixed point.

In the next example we have a rectangular metric (X,d) which is not Hausdorff and a mapping $T : X \to X$ which even after satisfying all the conditions of our above Theorem do not have fixed point.

Example 3.4. Let $A = \{0,2\}$, $B = \{\frac{1}{n} : n \in \mathbb{N}\}$, $X = A \cup B$. Define $d : X \times X \to [0,\infty)$ as follows

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \text{ and } \{x,y\} \subset A \text{ or } \{x,y\} \subset B \\ y, & x \in A,\ y \in B \\ x, & x \in B,\ y \in A \end{cases}$$

Then (X,d) is a generalized metric space which is complete. And (X,d) is not Hausdorff as there is no $r > 0$ such that $B(0,r) \cap B(2,r) = \varphi$. Let $\alpha, \eta : X \times X \to [0,\infty)$ by

$$\alpha(x,y) = \begin{cases} 3, & x \neq 0,\ y \neq \frac{1}{n} \\ 1, & x = 0,\ y = \frac{1}{n} \end{cases}$$

$$\eta(x,y) = 2 \text{ for all } x,y \in X \times X.$$

Define $T : X \to X$ as

$$T(0) = \frac{1}{2},\quad T(2) = 0,\quad T\left(\frac{1}{n}\right) = 0.$$

Using the notations as before we have

	$x=0,\ y=2$ $Tx=\frac{1}{2},\ Ty=0$	$x=0,\ y=\frac{1}{n}$ $Tx=\frac{1}{2},\ Ty=0$	$x=2,\ y=\frac{1}{n}$ $Tx=0,\ Ty=0$	$x=\frac{1}{m},\ y=\frac{1}{n}$ $Tx=0,\ Ty=0$
A	1	1/n	1/n	1
B	0.5	0.5	1	1/m
C	1	1/n	1/n	1/n
D	0.25	1/2(n+1)	1/n+1	1/2mn
E	0.33	1/3n	1/n	1/2mn
$d(Ty,Ty)$	0.5	0.5	0	0
$M(x,y)$	1	< 0.5, if $n > 2$	1	1

If we define $\psi : [0,\infty) \to [0,\infty)$ as $\psi(t) = \frac{3}{4}t$ then from the above table it is clear that $d(Tx,Ty) \leq \psi(M(x,y))$ holds whenever $\alpha(x,y) \geq \eta(x,y)$. But T has no fixed point.

Example 3.5. Let $X = \{1,3,5,7\}$ and $T : X \to X$ be defined as

$$T(1) = 3, T(3) = 5, T(5) = 7, T(7) = 1.$$

Let $\alpha, \eta : X \times X \to [0,\infty)$ given by

$$\alpha(x,y) = \begin{cases} 3, & x \neq 5,\ y \neq 7 \\ 2, & x = 5,\ y = 7 \end{cases}$$

$$\eta(x,y) = \begin{cases} 1, & x \neq 5,\ y \neq 7 \\ 3, & x = 5,\ y = 7 \end{cases}$$

Let $d : X \times X \to [0,\infty)$ be defined as

$$d(x,y) = d(y,x) \text{ and } d(x,y) = 0 \text{ iff } x = y$$

$$d(1,3) = 3, \quad d(1,5) = 1, \quad d(1,7) = 4, \quad d(3,5) = 2, \quad d(3,7) = 4, \quad d(5,7) = 1.$$

Then (X,d) is rectangular metric space. Also it is Hausdorff.

	$x=1, y=3$ $Tx=3, Ty=5$	$x=1, y=5$ $Tx=3, Ty=7$	$x=1, y=7$ $Tx=3, Ty=1$	$x=3, y=5$ $Tx=5, Ty=7$	$x=3, y=7$ $Tx=5, Ty=1$
A	3	1	4	2	4
B	3	3	3	2	2
C	2	5	3	1	1
D	1.5	7.5	1.8	0.66	0.4
E	2	3	4.25	1	1
$d(Ty,Ty)$	2	4	3	1	1
$M(x,y)$	3	7.5	4	2	4

For $x = 5$, $y = 7$, we have $d(Tx,Ty) = 4$ and $M(x,y) = 4$.
If we define $\psi : [0,\infty) \to [0,\infty)$ as $\psi(t) = \frac{4}{5}t$ then from the above table it is clear that $d(Tx,Ty) \leq \psi(M(x,y))$ holds whenever $\alpha(x,y) \geq \eta(x,y)$. But here for the periodic point $a = 5$, the condition $\alpha(a,Ta) \geq \eta(a,Ta)$ is not satisfied. As a result T has no fixed point.

If we take $\eta(x,y) = 1$ for $x,y \in X$ we have following corollary

Corollary 3.6. Let (X,d) be a Hausdorff and complete RMS, and let $T : X \to X$ be an $\alpha-$admissible mapping. Assume that there exists continuous function $\psi \in \Psi$ such that

$$x,y \in X,\ \alpha(x,y) \geq 1 \text{ implies } d(Tx,Ty) \leq \psi(M(x,y)). \tag{3.18}$$

where

$$M(x,y) = \max \left\{ \begin{array}{c} d(x,y), d(x,Tx), d(y,Ty), \\ \frac{1}{1+d(x,y)}[d(x,Tx)d(y,Ty)], \frac{1}{1+d(Tx,Ty)}[d(x,Tx)d(y,Ty)] \end{array} \right\}.$$

Also, suppose that the following assertions are hold;

1. there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$,
2. for all $x, y, z \in X$, $\alpha(x, y) \geq 1$ and $\alpha(y, z) \geq 1$ implies $\alpha(x, z) \geq 1$
3. either T is continuous or if $\{x_n\}$ is any sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$ then $\alpha(x_n, x) \geq 1$.

Then T has a periodic point $a \in X$ and if $\alpha(a, Ta) \geq 1$ holds then T has a fixed point. Moreover, if for all $x, y \in F(T)$, we have $\alpha(x, y) \geq 1$, then the fixed point is unique.

If we take $\alpha(x, y) = 1$ for $x, y \in X$ we have following corollary

Corollary 3.7. Let (X, d) be a Hausdorff and complete RMS, and let $T : X \to X$ be an η-admissible mapping. Assume that there exists continuous $\psi \in \Psi$ such that

$$x, y \in X,\ \eta(x, y) \leq 1 \text{ implies } d(Tx, Ty) \leq \psi(M(x, y)). \tag{3.19}$$

where

$$M(x,y) = \max\left\{ \begin{array}{c} d(x,y), d(x,Tx), d(y,Ty), \\ \frac{1}{1+d(x,y)}[d(x,Tx)d(y,Ty)], \frac{1}{1+d(Tx,Ty)}[d(x,Tx)d(y,Ty)] \end{array} \right\}.$$

Also, suppose that the following assertions are hold;

1. there exists $x_0 \in X$ such that $1 \geq \eta(x_0, Tx_0)$,
2. for all $x, y, z \in X$, $\alpha(x, y) \geq \eta(x, y)$ and $\alpha(y, z) \geq \eta(y, z)$ implies $1 \geq \eta(x, z)$
3. either T is continuous or for any sequence $\{x_n\}$ be any sequence in X with $1 \geq \eta(x_n, x_{n+1})$, for all $n \geq N$ and $x_n \to x$ as $n \to \infty$, implies that $1 \geq \eta(x_n, x)$, for all $n \geq N$.

Then T has a periodic point $a \in X$ and if $1 \geq \eta(a, Ta)$ holds then T has a fixed point. Moreover, if for all $x, y \in F(T)$, we have $1 \geq \eta(x, y)$, then the fixed point is unique.

If we take $\psi(t) = kt$ where $0 < k < 1$ then we have

Corollary 3.8. Let (X, d) be a Hausdorff and complete RMS, and let $T : X \to X$ be an $\alpha-$admissible mapping with respect to η. Assume that

$$x, y \in X, \alpha(x, y) \geq \eta(x, y) \text{ implies } d(Tx, Ty) \leq kM(x, y). \tag{3.20}$$

where

$$M(x,y) = \max\left\{ \begin{array}{c} d(x,y), d(x,Tx), d(y,Ty), \\ \frac{1}{1+d(x,y)}[d(x,Tx)d(y,Ty)], \frac{1}{1+d(Tx,Ty)}[d(x,Tx)d(y,Ty)] \end{array} \right\}.$$

Also, suppose that the following assertions are hold;

1. there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$,
2. for all $x, y, z \in X$, $\alpha(x, y) \geq \eta(x, y)$ and $\alpha(y, z) \geq \eta(y, z)$ implies $\alpha(x, z) \geq \eta(x, z)$
3. either T is continuous or if $\{x_n\}$ is any sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x$ as $n \to \infty$ then $\alpha(x_n, x) \geq 1$.

Then T has a periodic point $a \in X$ and if $\alpha(a, Ta) \geq \eta(a, Ta)$ holds then T has a fixed point. Moreover, if for all $x, y \in F(T)$, we have $\alpha(x, y) \geq \eta(x, y)$, then the fixed point is unique.

4 Application to Fractional Calculus

Here we give an application of Corollary 1 which guarantees the existence of solution for a nonlinear fractional differential equation considered in [2] (see also([5]). We will study the existence of solutions for the nonlinear fractional differential equation:

$$^{C}D^{\beta}(x(t)) = f(t, x(t)) \qquad (0 < t < 1,\ 1 < \beta \leq 2) \tag{4.1}$$

via the integral boundary conditions

$$x(0) = 0, \quad x(1) = \int_0^{\eta} x(s)ds, \quad (0 < \eta < 1)$$

where $^{C}D^{\beta}$ denotes the Caputo fractional derivative of order β and $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a continuous function. Consider the space $X = C([0,1],\mathbb{R})$, which is Banach space with supremum norm $\| x \|_{\infty} = \sup_{t\in[0,1]} |x(t)|$. We know that for a continuous function $g : [0,\infty) \to \mathbb{R}$, the Caputo derivative of fractional order β is defined as

$$^{C}D^{\beta}(g(t)) = \frac{1}{\Gamma(n-\beta)} \int_0^t (t-s)^{n-\beta-1} g^{n}(s)ds, \qquad (n-1 < \beta < n, n = [\beta]+1)$$

where $[\beta]$ denotes the integer part of the real number β and Γ is a gamma function.

Now, we prove the following existence theorem.

Theorem 4.1. Let $\xi : \mathbb{R}\times\mathbb{R} \to [0,\infty)$ be given function. Suppose that following conditions hold:

(i) there exists $\psi \in \Psi$ such that $|f(t,a) - f(t,b)| \leq \frac{\Gamma(\beta+1)}{5}\psi(|a-b|)$ for all $t \in [0,1]$ and $a, b \in \mathbb{R}$ with $\xi(a,b) \geq 0$,

(ii) there exists $x_0 \in X$ such that $\xi(x_0(t), Tx_0(t)) \geq 0$ for all $t \in [0,1]$, where the operator $T : X \to X$ is defined by

$$Tx(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,x(s))ds - \frac{2t}{(2-\eta^2)\Gamma\beta} \int_0^1 (1-s)^{\beta-1} f(s,x(s))ds$$
$$+\frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^{\eta} \left(\int_0^s (s-m)^{\beta-1} f(m,x(m))dm \right) ds, \quad (t \in [0,1]),$$

(iii) for each $t \in [0,1]$ and $x, y \in X, \xi(x(t), y(t)) \geq 0$ implies $\xi(Tx(t), Ty(t)) \geq 0$,

(iv) if $\{x_n\}$ is a sequence in X such that $x_n \to x$ in X and $\xi(x_n, x_{n+1}) \geq 0$ for all $n \in \mathbb{N}$, then $\xi(x_n, x) \geq 0$ for all $n \in \mathbb{N}$.

Then, the problem (4.1) has at least one solution.

Proof. It is well known that $x \in X$ is a solution of (4.1) iff $x \in X$ is a solution of the integral equation

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,x(s))ds - \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f(s,\beta(s))ds \\ &+ \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s-m)^{\beta-1} f(m,x(m))dm \right) ds, \quad t \in [0,1]. \end{aligned}$$

Then, the problem (4.1) is equivalent to find $x^* \in X$ which is a fixed point of T.

Now, let $x, y \in X$ such that $\xi(x(t), y(t)) \geq 0$ for all $t \in [0,1]$. By (i), we have

$$\begin{aligned} |Tx(t) - Ty(t)| &= \Bigg| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,x(s))ds \\ &- \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f(s,x(s))ds \\ &+ \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s-m)^{\beta-1} f(m,x(m))dm \right) ds \\ &- \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s,y(s))ds \\ &+ \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} f(s,y(s))ds \\ &- \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left(\int_0^s (s-m)^{\beta-1} f(m,y(m))dm \right) ds \Bigg| \\ &\leq \frac{1}{\Gamma(\beta)} \int_0^t |t-s|^{\beta-1} |f(s,x(s)) - f(s,y(s))|ds \\ &+ \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1} |f(s,x(s)) - f(s,y(s))|ds \\ &+ \frac{2t}{(2-\eta^2)\Gamma(\beta)} \int_0^\eta \left| \int_0^s (s-m)^{\beta-1} (f(m,y(m)) - f(m,x(m)))dm \right| ds \end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{\Gamma(\beta)}\int_0^{\eta}|t-s|^{\beta-1}\frac{\Gamma(\beta+1)}{5}\psi((|x(s)-y(s)|))ds \\
&+\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^{\eta}(1-s)^{\beta-1}\frac{\Gamma(\beta+1)}{5}\psi(|y(s)-x(s)|)ds \\
&+\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^{\eta}\left(\int_0^{s}|s-m|^{\beta-1}\frac{\Gamma(\beta+1)}{5}\psi(|y(m)-x(m)|)dm\right)ds \\
&\leq \frac{\Gamma(\beta+1)}{5}\psi(\|x-y\|_\infty)\sup\Big(\int_0^1|t-s|^{\beta-1}ds \\
&+\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^1(1-s)^{\beta-1}ds+\frac{2t}{(2-\eta^2)\Gamma(\beta)}\int_0^{\eta}\int_0^{s}|s-m|^{\beta-1}dmds\Big) \\
&\leq \psi(\|x-y\|_\infty).
\end{aligned}$$

This implies that for each $x, y \in X$ with $\xi(x(t), y(t)) \geq 0$ for all $t \in [0,1]$, we obtain that

$$\|Tx-Ty\|_\infty \leq \psi(\|x-y\|_\infty) \tag{4.2}$$

Now, we define function $\alpha : X \times X \to [0,\infty)$ by

$$\alpha(x,y)=\begin{cases} 1, & \xi(x(t),y(t)) \geq 0 \text{ for all } t \in [0,1]; \\ 0, & \text{otherwise} \end{cases}$$

Then, we obtain that

$$d(Tx,Ty) \leq \psi(d(x,y))$$

for all $x, y \in X$.
Now, by using condition (iii), we get

$$\alpha(x,y) \geq \eta(x,y) \Rightarrow \xi(x(t),y(t)) \geq 0 \Rightarrow \xi(Tx(t),Ty(t)) \geq 0 \Rightarrow \alpha(Tx,Ty) \geq \eta(Ty,Ty)$$

for all $x, y \in X$ and hence T is α-admissible mapping with respect to η. Also, from (ii), there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq \eta(x_0, Tx_0)$. Finally, from (iv) we get easily that condition (iii) of Corollary 1 holds true. Thus, as an application of our Corollary 1 , we deduce the existence of $x^* \in X$ such that $x^* = Tx^*$ and hence x^* is a solution of the problem (4.1). Q.E.D.

5 Conclusion

In the area of fractional calculus, the study of fractional nonlinear differential equations and their applications have become a topic of high interest because it has been widely applied to resolve various problems that arise in several branches of science and engineering. As a consequence, many mathematicians tried to study this methods. In this paper, based on the present new concept of $\alpha - \psi$ type contractive mapping in rectangular metric space, we have studied fixed point results for such mappings and proved the existence theorems for nonlinear fractional differential equations for the integral boundary conditions.

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